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Invariant constituents and invariant blocks under coprime action

Ziqun Lu

School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

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Abstract

Let A and G be finite groups with $(|A|, |G|) = 1$. We assume that A acts on G via automorphism. Let N be an A-invariant normal subgroup of G. Let φ be an A-invariant irreducible Brauer character of N. If A is of prime power order, then the induced Brauer character φ^G contains an A-invariant irreducible constituent; If G/N is p-solvable, then φ^G contains an A-invariant irreducible constituent. Let B be an A-invariant block of G. Then under Glauberman–Isaacs correspondence, the set $\mathrm{Irr}_A(B)$ is a union of blocks of $C_G(A)$, say $b_1, b_2, \ldots, b_s$. Let Q_i be a defect group of b_i. Then there is a defect group D of B such that $Q_i \leqslant D$.

1. Introduction

Let G be a finite group. Let (K, R, F) be a p-modular system, where R is a complete discrete valuation ring with a unique maximal ideal (π) for $\pi \in R$, K is the quotient field of R with characteristic zero and $F = R/(\pi)$ is an algebraically closed field with characteristic $p > 0$. We fix a valuation ν of K such that R is its valuation ring and $\nu(\pi) = 1$. For an RG-module (or FG-module) V, we denote by $\mathrm{hd}(V)$ (respectively $\mathrm{soc}(V)$) the head (respectively the socle) of V.

Let A be a finite group such that A acts on G and $(|A|, |G|) = 1$, where $|A|$ and $|G|$ denote the orders of A and G, respectively. We denote by $\mathrm{Irr}(G)$ (respectively $\mathrm{Irr}_A(G)$) the set of irreducible ordinary characters (respectively the set of A-invariant irreducible ordinary characters) of G. When A is solvable, Glauberman defines a one-to-

E-mail address: lu@g.math.s.chiba-u.ac.jp.

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one correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(C_G(A))$. When $|G|$ is odd, Isaacs also defines a one-to-one correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(C_G(A))$. Wolf [14] proves that these two correspondences are the same if $|G|$ is odd and A is solvable. We let $*$ denote the Glauberman–Isaacs correspondence, thus $\chi \mapsto \chi^*$ is a one-to-one correspondence from $\mathrm{Irr}_A(G)$ to $\mathrm{Irr}(C_G(A))$. Let B be an A-invariant block of RG. Denote by $\mathrm{Irr}_A(B)$ the set of A-invariant irreducible ordinary characters in B.

We assume in the rest of this paper that A and G are of coprime orders and A acts through automorphisms on G. In this paper, a module means a finitely generated right module. For a subgroup H of G, and for an FG-module X and an FH-module Y, we write X_H for the restriction of X to H and Y^G for the induction of Y to G. When $H \lhd G$ and Y is an FH-module, we denote $I_G(Y)$ the inertia subgroup of Y in G.

2. Stable constituents under coprime action

Let H be an A-invariant subgroup of G. Let φ be an A-invariant irreducible Brauer character of H. In this section, we prove that there is an A-invariant irreducible Brauer character as a constituent in φ^G under some assumption.

Proposition 1. *Assume that A is of prime power order, say q^n $(q \neq p)$. Let H be an A-invariant subgroup of G. Let $\varphi \in \mathrm{IBr}_A(H)$ with $q \nmid \varphi(1)$, then there exists an A-invariant irreducible constituent in φ^G. Specially, if H is an A-invariant p-solvable subgroup of G and $\varphi \in \mathrm{IBr}_A(H)$, then there exists an A-invariant irreducible constituent in φ^G.*

Proof. Set $\varphi^G = \sum_{\beta \in \mathrm{IBr}(G)} m_\beta \beta$. Let $S = \{\beta \in \mathrm{IBr}(G) \mid m_\beta \neq 0\}$. Then A permutes the elements in S. Let $O_1, O_2, \ldots, O_n$ be all the A-orbits of A on S. Let $\beta_i \in O_i$. If irreducible Brauer characters β and η are in the same orbit, then $m_\beta = m_\eta$. Thus $\varphi^G(1) = \sum_{i=1}^n |A : C_A(\beta_i)| m_{\beta_i} \beta_i(1)$. Since $q \nmid \varphi^G(1) = |G : H|\varphi(1)$, there exists an i such that $A = C_A(\beta_i)$. So β_i is an A-invariant irreducible constituent of φ^G, as desired. $\square$

Let N be a normal subgroup of G, and let W be an indecomposable FN-module. We assume that $I_G(W) = G$. Set $E = \mathrm{End}_{FG}(W^G)$ and $\Lambda = \mathrm{End}_{FN}(W)$. We can write E in the form $E = \bigoplus_{\bar{y} \in Y} E_{\bar{y}}$ where $Y = G/N$ and $E_{\bar{y}}$ is the F-submodule of E mapping $W = W \otimes 1$ to $W \otimes y$ inside W^G, and $E_{\bar{x}} \cong \mathrm{Hom}_{FN}(W, Wx)$ as F-module by [10, 4.6.4]. Clearly $E_{\bar{x}} E_{\bar{y}} \subset E_{\overline{xy}}$, for $\bar{x}, \bar{y} \in Y$. Also we can use the stability hypothesis to choose an element $\varphi_{\bar{y}} \in E_{\bar{y}}$ mapping $W \otimes 1$ isomorphically onto $W \otimes y$; it follows that $\varphi_{\bar{y}}$ is a unit in E. Since $E_{\bar{1}}$ can be identified with Λ, we have $E_{\bar{y}} = \Lambda \varphi_{\bar{y}} = \varphi_{\bar{y}} \Lambda$. So E is a free right Λ-module. The module $E \otimes_\Lambda W$ is an E–FG-bimodule with actions $(e \otimes w) \cdot y := e\varphi_{\bar{y}} \otimes \varphi_{\bar{y}}^{-1}(w \otimes y)$, where $\bar{y} = yN$, and $e' \cdot (e \otimes w) := e'e \otimes w$. Then we have the following proposition due to Cline, see [10, 4.6.6].

Proposition 2. *There is an E–FG-bimodule isomorphism $E \otimes_\Lambda W \cong W^G$ given by $f : e \otimes w \mapsto e(w)$, for $e \in E$ and $w \in W$.*

Proposition 3 [8, Corollary 1.2]. *Keep the notations as above. Let $E = \bigoplus U_i$ be a decomposition into indecomposable E-modules. Then $W^G = \bigoplus U_i W \cong \bigoplus U_i \otimes_\Lambda W$ is a decomposition into indecomposable FG-modules. Moreover we have that* $\dim(U_i W) = \operatorname{rank}_\Lambda(U_i)\dim(W)$, *and that $U_i = U_j$ as E-modules if and only if $U_i W \cong U_j W$ as FG-modules.*

The following result is essentially due to Harris (see [5, Theorem 7]), but we also give a proof here for convenience to readers.

Proposition 4. *Let N be a normal subgroup of G, and let W be an irreducible FN-module. Then for any indecomposable direct summand V of W^G,* $\mathrm{hd}(V)$ *and* $\mathrm{soc}(V)$ *are irreducible. If P is a projective cover of W, then P^G is a projective cover of W^G.*

Proof. By induction on $|G/N|$, we can assume that $I_G(W) = G$. Since P is a projective cover of W, then $I_G(P) = I_G(W) = G$. We may assume that $W = \mathrm{soc}(P)$. Set $E = \mathrm{End}_{FG}(P^G)$ and $\Lambda = \mathrm{End}_{FN}(P)$. Set $Y = G/N$. We can write E in the form $E = \bigoplus_{\bar{y}\in Y} E_{\bar{y}}$. We identify $E_{\bar{1}}$ with Λ and let $\varphi_{\bar{y}}$ be an invertible element in $E_{\bar{y}}$. Then $E_{\bar{y}} = \Lambda\varphi_{\bar{y}} = \varphi_{\bar{y}}\Lambda$. Thus $E/J(\Lambda)E \cong \bigoplus_{\bar{y}\in Y} \varphi_{\bar{y}} F$ is a twisted group algebra of G/N over F. We let $\varphi_{\bar{y}}|_{W^G} = \varphi'_{\bar{y}}$, it is obvious that $\varphi'_{\bar{y}}$ is a unit in $\mathrm{End}_{FG}(W^G)$ mapping $W \otimes 1$ isomorphically onto $W \otimes y$. Thus $\mathrm{End}_{FG}(W^G) \cong \bigoplus_{\bar{y}\in Y} \varphi'_{\bar{y}} F$ is a twisted group algebra of G/N over F. It is easy to verify that $\mathrm{End}_{FG}(W^G)$ is isomorphic to $E/J(\Lambda)E$. Suppose that $E = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ (respectively $\mathrm{End}_{FG}(W^G) = V'_1 \oplus V'_2 \oplus \cdots \oplus V'_n$) is a decomposition into indecomposable E- (respectively $\mathrm{End}_{FG}(W^G)$-) modules. By Proposition 3, $P^G = V_1 P \oplus \cdots \oplus V_m P$ (respectively $W^G = V'_1 W \oplus \cdots \oplus V'_n W$) is a decomposition into indecomposable FG-modules. Since $E/J(\Lambda)E \cong \mathrm{End}_{FG}(W^G)$ and $J(\Lambda)E \leqslant J(E)$, we have $m = n$. Since there is a surjective FG-module homomorphism from P^G to W^G, we must have that the head of $V'_i W$ is irreducible for $i = 1, 2, \ldots, n$ and P^G is a projective cover of W^G.

Let W^* be the dual of W, thus each indecomposable direct summand of $(W^*)^G$ has irreducible head. Since $(W^*)^G \cong (W^G)^*$, each indecomposable direct summand of W^G has irreducible socle, as desired. □

Theorem A. *Assume that A is of prime power order. Let N be an A-invariant normal subgroup of G, and let φ be an A-invariant irreducible Brauer character of N. Then there exists some A-invariant irreducible constituent in φ^G.*

Proof. We can assume that φ is G-invariant by induction on $|G/N|$. Let W be an irreducible FN-module such that W provides Brauer character φ. We denote by NA the semidirect product of N and A. Since $(|A|, |N|) = 1$, W can be extended to an $F(NA)$-module. Thus we can view W as an FA-module. Thus $E = \mathrm{End}_{FG}(W^G)$ becomes an FA-module with the action defined by

$$(e \cdot a)(v) := e\bigl(va^{-1}\bigr)a \quad \text{for } e \in E,\ a \in A,\ \text{and } v \in W^G.$$

Let $L_1, \ldots, L_n$ be a complete set of non-isomorphic irreducible E-modules, and let P_i be the projective cover of L_i. Since E is an FA-module, A permutes the set $\{P_1, \ldots, P_n\}$. Since $E \cong \bigoplus_{i=1}^n (\dim L_i) P_i$ and $\dim E = |G/N|$ is coprime to $|A|$, we have that one of P_i must be A-invariant. Thus by Proposition 3, $P_i W \cong P_i \otimes_F W$ is an indecomposable direct summand of W^G and A-invariant. Let V be the head of $P_i W$. Then V is irreducible by Proposition 4. Thus V is an A-invariant irreducible constituent of W^G, as desired. $\square$

Theorem B. *Let N be an A-invariant normal subgroup of G. Let $\varphi \in \mathrm{IBr}_A(N)$. If G/N is p-solvable, there exists an A-invariant irreducible constituent in φ^G.*

Proof. We denote by GA the semidirect product of G and A. Let W be an irreducible FN-module such that W provides Brauer character φ. By induction on $|G/N|$, we can assume that G/N is a principal factor of GA. Since G/N is p-solvable, G/N is a p-group or a p'-group.

If G/N is a p-group, then W^G is an indecomposable FG-module. Let V be the head of W^G. By Proposition 4, V is irreducible, and consequently V is A-invariant. Thus V is an A-invariant irreducible constituent of W^G, as desired.

We assume now that G/N is a p'-group. By induction on $|G/N|$ again, we may assume that φ is G-invariant. Then $I_G(W) = G$. Set $E = \mathrm{End}_{FG}(W^G)$ and $\Lambda = \mathrm{End}_{FN}(W) \cong F$. Let X be a complete set of right coset representatives of N in G. Then $E \cong \bigoplus_{x\in X} \varphi_{\bar{x}} F$, where $\varphi_{\bar{x}}$ is a unit in E mapping $W \otimes 1$ isomorphically onto $W \otimes x$ and $\varphi_{\bar{1}} = 1$. Then E is isomorphic to a twisted group algebra of G/N over F with factor set α, write $F_\alpha G/N$. Since $H^2(G/N, F^\times)$ is a finite group of exponent which is a factor of $|G/N|$, we have $\alpha^{|G/N|} \sim 1$. Then there exists a map $\eta : G/N \to F^\times$ such that $\alpha(\bar{x}, \bar{y})^{|G/N|} = \eta(\bar{x})\eta(\bar{y})\eta(\bar{x}\bar{y})^{-1}$ for $\bar{x}, \bar{y} \in G/N$. Let $k(\bar{x})$ be a $|G/N|$th root of $\eta(\bar{x})$ in $F^\times$. Set $\varphi'_{\bar{x}} = k(\bar{x})^{-1}\varphi_{\bar{x}}$ and $\varphi'_{\overline{x_1}}\varphi'_{\overline{x_2}} = \alpha(\overline{x_1}, \overline{x_2})'\varphi'_{\overline{x_1 x_2}}$ for $\alpha(\overline{x_1}, \overline{x_2})' \in F^\times$. It is easy to see that $\varphi'_{\bar{x}} \in E_{\bar{x}}$, $\alpha(\overline{x_1}, \overline{x_2})'$ is a factor set of G/N and $\alpha(\overline{x_1}, \overline{x_2})' = k(\overline{x_1})^{-1}k(\overline{x_2})^{-1}k(\overline{x_1}\overline{x_2})\alpha(\overline{x_1}, \overline{x_2})$. Thus $(\alpha(\overline{x_1}, \overline{x_2})')^{|G/N|} = 1$. Since $E \cong \bigoplus_{x\in X} F\varphi_{\bar{x}} = \bigoplus_{x\in X} F\varphi'_{\bar{x}}$, we can assume that $\alpha(\overline{x_1}, \overline{x_2})^{|G/N|} = 1$. From now on, we assume that $\alpha^{|G/N|} = 1$.

We can view W as an FA-module. Then W^G and $E = \mathrm{End}_{FG}(W^G)$ are FA-modules with actions defined respectively by

$$\left(\sum_{x\in X} w_x \otimes x\right) \cdot a := \sum_{x\in X} (w_x)a \otimes x^a \quad \text{for } w_x \in W \text{ and } a \in A,$$

and

$$(e \cdot a)(v) := e\bigl(va^{-1}\bigr)a \quad \text{for } e \in E,\ a \in A,\ \text{and } w \in W^G.$$

Thus $E \otimes_F W$ is an FA-module with the action

$$(e \otimes w) \cdot a := e \cdot a \otimes wa \quad \text{for } e \in E,\ w \in W,\ \text{and } a \in A.$$

By Proposition 2, $f : e \otimes w \mapsto e(w)$ is an isomorphism from $E \otimes_F W$ to W^G. It is easy to see that f is also an FA-module isomorphism. For $a \in A$, we assume that

$\varphi_{\bar{y}} \cdot a = k_{\bar{y}}^a \varphi_{\overline{y^a}}$, where $k_{\bar{y}}^a \in F^\times$ is determined by $\bar{y}$ and a. Since $(\varphi_{\bar{x}}\varphi_{\bar{y}}) \cdot a = (\varphi_{\bar{x}} \cdot a)(\varphi_{\bar{y}} \cdot a)$ and $(\varphi_{\bar{x}}) \cdot a_1 a_2 = (\varphi_{\bar{x}} \cdot a_1) \cdot a_2$ for $a, a_1, a_2 \in A$, we have

$$\alpha(\bar{x}, \bar{y}) k_{\bar{x}\bar{y}}^a = \alpha\bigl(\overline{x^a}, \overline{y^a}\bigr) k_{\bar{x}}^a k_{\bar{y}}^a \quad \text{and} \quad k_{\bar{x}}^{a_1 a_2} = k_{\bar{x}}^{a_1} k_{\overline{x^{a_1}}}^{a_2}.$$

Let $\bar{x}$ be an element of G/N of order r. For $a \in A$, we have $(\varphi_{\bar{x}})^r \cdot a = (k_{\bar{x}}^a)^r (\varphi_{\overline{x^a}})^r$. Since

$$(\varphi_{\bar{x}})^r \cdot a = \alpha(\bar{x}, \bar{x}) \alpha\bigl(\bar{x}^2, \bar{x}\bigr) \cdots \alpha\bigl(\bar{x}^{r-1}, \bar{x}\bigr) 1_E \cdot a = \alpha(\bar{x}, \bar{x}) \alpha\bigl(\bar{x}^2, \bar{x}\bigr) \cdots \alpha\bigl(\bar{x}^{r-1}, \bar{x}\bigr) 1_E$$

and

$$\bigl(k_{\bar{x}}^a\bigr)^r (\varphi_{\overline{x^a}})^r = \bigl(k_{\bar{x}}^a\bigr)^r \alpha\bigl(\overline{x^a}, \overline{x^a}\bigr) \alpha\bigl(\bigl(\overline{x^a}\bigr)^2, \overline{x^a}\bigr) \cdots \alpha\bigl(\bigl(\overline{x^a}\bigr)^{r-1}, \overline{x^a}\bigr) 1_E,$$

we have

$$\bigl(k_{\bar{x}}^a\bigr)^r \alpha\bigl(\overline{x^a}, \overline{x^a}\bigr) \alpha\bigl(\bigl(\overline{x^a}\bigr)^2, \overline{x^a}\bigr) \cdots \alpha\bigl(\bigl(\overline{x^a}\bigr)^{r-1}, \overline{x^a}\bigr) = \alpha(\bar{x}, \bar{x}) \alpha\bigl(\bar{x}^2, \bar{x}\bigr) \cdots \alpha\bigl(\bar{x}^{r-1}, \bar{x}\bigr).$$

Then $\alpha^{|G/N|} = 1$ implies $(k_{\bar{x}}^a)^{|G/N|r} = 1$. Thus $k_{\bar{x}}^a$ is of finite order and coprime to $|A|$.

Let Z be a finite group generated by $\{\alpha(\bar{x}, \bar{y}), k_{\bar{x}}^a \mid \bar{x}, \bar{y} \in \overline{G},\ a \in A\}$. Thus the set $G^* = Z \times \overline{G} = \{(z, \bar{x}) \mid z \in Z,\ \bar{x} \in \overline{G}\}$ is a group with the multiplication defined by

$$(z, \bar{x})\bigl(z', \bar{y}\bigr) = \bigl(\alpha(\bar{x}, \bar{y}) z z', \bar{x}\bar{y}\bigr).$$

We define an action of A on G^* by

$$(z, \bar{x})^a = \bigl(z k_{\bar{x}}^a, \bar{x}^a\bigr), \quad \text{for } a \in A,\ \bar{x} \in \overline{G}, \text{ and } z \in Z.$$

We claim that this action is a group action. Since

$$\bigl((z_1, \overline{x_1})(z_2, \overline{x_2})\bigr) \cdot a = \bigl(\alpha(\overline{x_1}, \overline{x_2}) z_1 z_2, \overline{x_1 x_2}\bigr) \cdot a = \bigl(\alpha(\overline{x_1}, \overline{x_2}) z_1 z_2 k_{\overline{x_1 x_2}}^a, \overline{(x_1 x_2)^a}\bigr)$$

and

$$\bigl((z_1, \overline{x_1}) \cdot a\bigr)\bigl((z_2, \overline{x_2}) \cdot a\bigr) = \bigl(z_1 k_{\overline{x_1}}^a, \overline{x_1^a}\bigr)\bigl(z_2 k_{\overline{x_2}}^a, \overline{x_2^a}\bigr) = \bigl(\alpha\bigl(\overline{x_1^a}, \overline{x_2^a}\bigr) z_1 z_2 k_{\overline{x_1}}^a k_{\overline{x_2}}^a, \overline{(x_1 x_2)^a}\bigr),$$

we have $((z_1, \overline{x_1})(z_2, \overline{x_2})) \cdot a = ((z_1, \overline{x_1}) \cdot a)((z_2, \overline{x_2}) \cdot a)$. By the same way, $(z, \bar{x}) \cdot a_1 a_2 = ((z, \bar{x}) \cdot a_1) \cdot a_2$. Thus the claim is correct.

Let $\lambda : Z \to F^\times$ $(z \mapsto z)$, a representation of Z. The primitive idempotent e_λ of FZ corresponding to λ is a central idempotent of FG^*. And the map

$$\rho : E = \bigoplus_{\bar{x} \in \overline{G}} F\varphi_{\bar{x}} \to e_\lambda FG^* \quad \bigl(\varphi_{\bar{x}} \mapsto e_\lambda(1, \bar{x})\bigr)$$

is an F-algebra isomorphism, and moreover ρ is an A-algebra isomorphism.

Since A acts trivially on Z, $e_\lambda FZ$ is an A-invariant irreducible FZ-module. Note that $p \nmid |G^*|$. Thus by [11, Theorem A], $(e_\lambda FZ)^{G^*} \cong e_\lambda FG^*$ has an A-invariant irreducible constituent, say U. Thus $f^{-1}(U)$ is an A-invariant irreducible constituent of E. Since $E \cong F_\alpha G/N$ is semisimple, $f^{-1}(U)$ is a direct summand of E. Set $V = f^{-1}(U) \otimes_F W$. Then V is an A-invariant irreducible constituent of W^G by Proposition 3, as desired. □

3. Invariant blocks under coprime action

G, A are as before.

Proposition 5 [7, Lemma 13.8 and Corollary 13.9]. *Assume that both A and G act on a set Ω and that G acts transitively on Ω. In addition, suppose that $(\omega g)a = (\omega a)g^a$ for all $a \in A$, $g \in G$, and $\omega \in \Omega$. Then*:

(a) *A fixes a point in Ω; and*
(b) *$C_G(A)$ acts transitively on the set of A-fixed points of Ω.*

In [3], Dade defined the following important subgroup of G.

Definition 1. Let G be a normal subgroup of a finite group Γ, and let B be a block of RG with block idempotent 1_B. Following Dade [3, p. 212], we define a subgroup $G[B]$ of Γ. Set $G[B] = \{x \in \Gamma \mid (1_B C_{\bar{x}})(1_B C_{\bar{x}^{-1}}) = 1_B C_{\bar{1}}\}$, where $C_{\bar{x}} = C_{R\Gamma}(G) \cap RGx$ for $x \in \Gamma$.

Let $C[B] = \bigoplus_{\sigma \in G[B]\backslash G} 1_B C_{\bar{\sigma}}$. Then $C[B]$ is a $G[B]/G$-graded Clifford system with $C[B]_{\bar{\sigma}} = 1_B C_{\bar{\sigma}}$. Since $C[B]_1/J(C[B]_1) \cong F$, $C[B]/(J(C[B]_1)C[B])$ is a twisted group algebra of $G[B]/G$ over F.

Proposition 6 (Dade [3, Theorem 3.7]). *There is a natural one-to-one correspondence between the blocks of Γ which cover B and the Γ/G-conjugacy classes of blocks of $C[B]/(J(C[B]_1)C[B])$.*

Proposition 7. *Assume that A is a cyclic group of prime order q. Let B be an A-invariant block of G. Then B is covered by q blocks or one block of GA. If B is covered by q blocks $\widetilde{B}_1, \ldots, \widetilde{B}_q$ of GA, then restriction is a one-to-one correspondence from* $\mathrm{Irr}(\widetilde{B}_i)$ *to* $\mathrm{Irr}(B)$.

Proof. Let Γ be the semi-direct product of G and A. Then $G[B]/G$ is of order q or 1. Thus by Proposition 6, B is covered by q blocks or 1 block of $\Gamma = GA$.

If B is covered by q blocks, then each irreducible character in B is GA-invariant. Let $\chi \in \mathrm{Irr}(B)$. Since $(|\Gamma : G|, |G|) = 1$ and $I_{GA}(\chi) = GA$, χ can be extended to GA. Let $\hat{\chi}$ be the canonical extension of χ to GA. Then $\chi^{GA} = \sum_{\lambda \in \mathrm{Irr}(GA/G)} \lambda\hat{\chi}$. Since each block $\widetilde{B}_i$ contains an irreducible constituent of χ^{GA}, $\widetilde{B}_i$ contains a unique irreducible constituent of χ^{GA} for each i. Thus restriction is a one-to-one correspondence from $\mathrm{Irr}(\widetilde{B}_i)$ to $\mathrm{Irr}(B)$. □

Remark. If B is covered by q blocks $\widetilde{B}_1, \ldots, \widetilde{B}_q$ of GA, then $\widetilde{B}_i$ and B are naturally Morita equivalent of degree 1 (cf. [6] or [9]). Thus B is isomorphic to $\widetilde{B}_i$ in the sense of Alperin [1] or Dade [4].

Theorem C. *Assume that B is an A-invariant block of G and that $\mathrm{Irr}_A(B)$ is not empty. Then, $\{\chi^* \mid \chi \in \mathrm{Irr}_A(B)\} = \mathrm{Irr}(b_1) \cup \cdots \cup \mathrm{Irr}(b_t)$ for some blocks $b_1, \ldots, b_t$ of $C_G(A)$.*

Proof. When G is solvable, the result is proved by Wolf [13, Theorem 4.8]. Thus we can assume that A is solvable. By induction on $|A|$, we can assume that A is a cyclic group of prime order q. Let a be a generator of A. Let χ be an A-invariant character of B. Then there exists a unique extension $\hat{\chi}$ of χ to GA and a sign $\varepsilon_\chi = \pm 1$ such that

$$\hat{\chi}(xc) = \varepsilon_\chi \chi^*(c)$$

for any $c \in C$ and any $1 \neq x \in A$, see [7, Theorem 13.6]. By Proposition 7, we have the following two cases.

Case 1. The block B is covered by q blocks of GA. Thus each irreducible character in B is A-invariant. Let $\widetilde{B}$ be one of the q blocks over B. Then $\widetilde{B}$ contains the same number of irreducible characters as B. Thus,

$$\mathrm{Irr}\big(\widetilde{B}\big) = \big\{\lambda_\chi \hat{\chi} \mid \chi \in \mathrm{Irr}(B) = \mathrm{Irr}_A(B),\ \text{for some } \lambda_\chi \in \mathrm{Irr}(GA/G)\big\}.$$

For any p-regular element $c \in G$ and any p-singular element $d \in C$, we have

$$\begin{aligned}
\sum_{\chi \in \mathrm{Irr}_A(B)} \chi^*\big(c^{-1}\big)\chi^*(d) &= \sum_{\chi \in \mathrm{Irr}_A(B)} \hat{\chi}\big(a^{-1}c^{-1}\big)\hat{\chi}(ad) \\
&= \sum_{\chi \in \mathrm{Irr}_A(B)} \big(\lambda_\chi \hat{\chi}\big)\big(a^{-1}c^{-1}\big)(\lambda_\chi \hat{\chi})(ad) \\
&= \sum_{\varphi \in \mathrm{Irr}(\widetilde{B})} \varphi\big(a^{-1}c^{-1}\big)\varphi(ad) = 0.
\end{aligned}$$

Thus by Osima [12, Theorem 3], $\{\chi^* \mid \chi \in \mathrm{Irr}_A(B)\}$ is a union of blocks of C, as desired.

Case 2. The block B is covered by one block $\widetilde{B}$ of GA. Thus, $\mathrm{Irr}(\widetilde{B}) = \{\lambda\hat{\chi} \mid \chi \in \mathrm{Irr}_A(B),\ \lambda \in \mathrm{Irr}(GA/G)\} \cup \{\chi^{GA} \mid \chi \in \mathrm{Irr}(G) \setminus \mathrm{Irr}_A(G)\}$. For any p-regular element $c \in C$ and any p-singular element $d \in C$, we have

$$\begin{aligned}
\sum_{\chi \in \mathrm{Irr}_A(B)} \chi^*\big(c^{-1}\big)\chi^*(d) &= \sum_{\chi \in \mathrm{Irr}_A(B)} \hat{\chi}\big(a^{-1}c^{-1}\big)\hat{\chi}(ad) \\
&= \frac{1}{q} \sum_{\lambda \in \mathrm{Irr}(GA/G)} \sum_{\chi \in \mathrm{Irr}_A(B)} \big(\lambda\hat{\chi}\big)\big(a^{-1}c^{-1}\big)\big(\lambda\hat{\chi}\big)(ad) \\
&\quad + \frac{1}{q^2} \sum_{\chi \in \mathrm{Irr}(B) \setminus \mathrm{Irr}_A(B)} \chi^{GA}\big(a^{-1}c^{-1}\big)\chi^{GA}(ad) \\
&= \frac{1}{q} \sum_{\varphi \in \mathrm{Irr}(\widetilde{B})} \varphi\big(a^{-1}c^{-1}\big)\varphi(ad) = 0.
\end{aligned}$$

Thus by Osima [12, Theorem 3] again, $\{\chi^* \mid \chi \in \mathrm{Irr}_A(B)\}$ is a union of blocks of C, as desired. □

Corollary to Theorem C. *Let $\chi_1, \chi_2 \in \mathrm{Irr}_A(G)$. If χ_1^* and χ_2^* are in the same block of C, then χ_1 and χ_2 are in the same block of G.*

Proposition 8 (Brauer [2, 3G]). *Let B be a block of G of defect group D. Let χ be an irreducible character of B and let σ be an element of G. If $\nu(\chi(\sigma)) = \alpha$, there exists a conjugate D^t of D for some $t \in G$ such that $|D^t \cap C_G(\sigma)| = p^\mu$ with $\mu \geqslant \nu(|C_G(\sigma)|) - \alpha$.*

Theorem D. *Let χ be an A-invariant irreducible character of G contained in a block B. Let b be the block of $C_G(A)$ containing χ^* with a defect group Q. Then there exists a defect group D of B such that $Q \subseteq_{C_G(A)} D \cap C_G(A)$.*

Proof. If G is p-solvable, it is proved by Wolf in [13, Theorem 4.9]. Thus we can assume that A is solvable. By induction on $|A|$, we can assume that A is of prime order q, and let a be a generator of A. Choose a height zero character χ_0^* in b. Then by the Corollary of Theorem C, χ_0 also belongs to B. Let C be a defect class of b and let $x \in C$. Then $\chi_0^*(x) \not\equiv 0 \pmod{\pi}$ by [10, Chapter 5 Theorem 1.11(ii)]. We can assume that Q is a Sylow p-subgroup of $C_{C_G(A)}(x)$ since Q is a defect group of b. Let $\hat{\chi}_0$ be the unique extension of χ_0 to GA, the semi-direct product of G and A. Since $\hat{\chi}_0(ax) = \varepsilon_{\chi_0}\chi_0^*(x)$, we have $\hat{\chi}_0(ax) \not\equiv 0 \pmod{\pi}$. Assume $\hat{\chi}_0$ belongs to a block $\widehat{B}$ of GA. By Proposition 8, there exists a defect group D of $\widehat{B}$ containing a Sylow p-subgroup of $C_{GA}(ax)$. Since $Q \subseteq C_{C_G(A)}(x) = C_G(A) \cap C_G(x) \subseteq C_{GA}(ax)$, D contains a conjugate Q^t of Q for some $t \in C_G(A)$. Since $I_{GA}(B) = GA$, $D = D \cap G$ is a defect group of B. Thus $Q \leqslant_{C_G(A)} D \cap C_G(A)$, as desired. □

Remark. Notations are as in Theorem D. By Proposition 5, B always has an A-invariant defect group. A further question is whether we can choose the defect group D of B in Theorem D to be A-invariant. It looks reasonable, but we do not find a way to prove it.

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